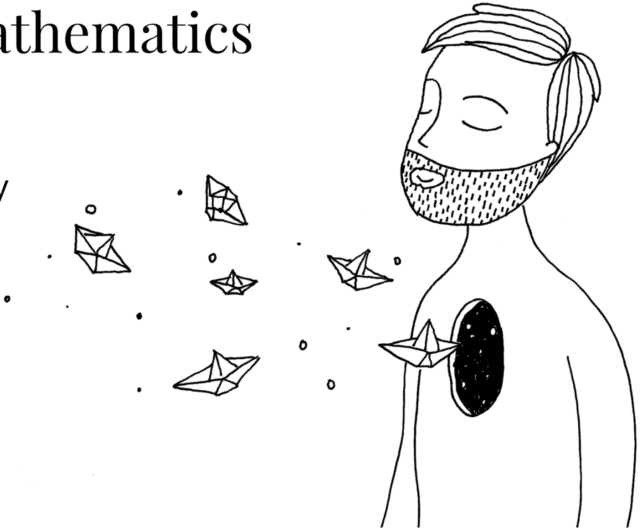
4509 - Bridging Mathematics

Concavity and Quasiconcavity



PAULO FAGANDINI

NOVA SCHOOL OF BUSINESS & ECONOMICS

Concavity

Definition (Concave and Convex function)

A real-valued function $f : \mathcal{U} \subseteq \mathbb{R}^n$ is **concave** if $\forall x, y \in \mathcal{U}$ and $\forall t \in [0, 1]$:

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$$

and it is **convex** if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

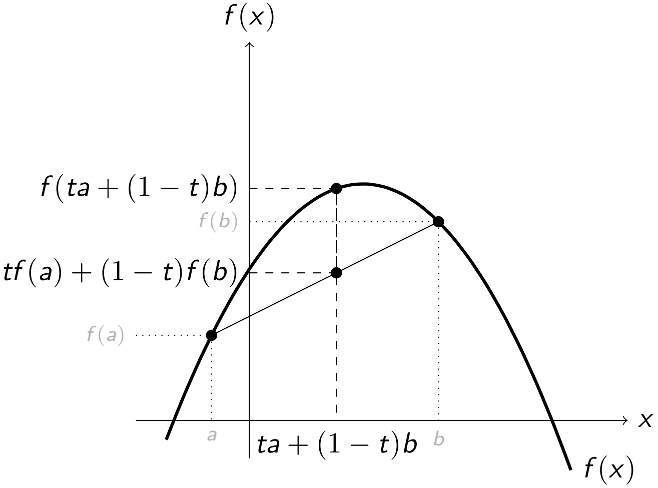
Concavity

Concavity and convexity of a function have also a geometric interpretation.

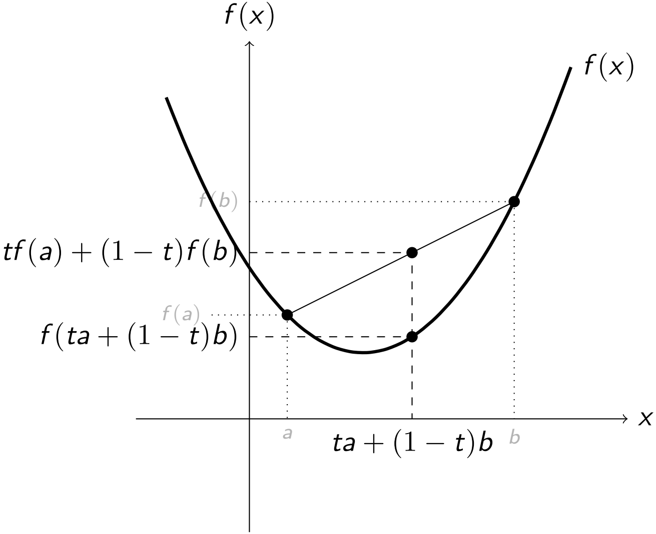
Proposition

A function $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is concave (convex)* $\Leftrightarrow \forall x, y \in \mathbb{R}^n$ *the secant line connecting* x *and* y *lies below (above) the graph of* f.

Concavity



Convexity



Concavity

Definition (Convex Set)

A set $\mathcal{U}$ is said to be *convex* if $\forall x, y \in \mathcal{U}$ and $\forall t \in [0, 1]$ it holds that:

$$tx + (1 - t)y \in \mathcal{U}$$

Facts:

1. It is not uncommon to call $tx + (1 - t)y$ with $t \in [0, 1]$ a *convex combination* between x and y.
2. A **concave set**, contrary to functions, is **not a thing**.

Concavity

Proposition

All convex or concave functions must have a convex domain.

Proposition

1. f *is concave if and only if the set below its graph* $\{(x, y) : y \leq f(x)\}$ *is convex.*
2. f *is convex if and only if the set above its graph* $\{(x, y) : y \geq f(x)\}$ *is convex.*

Concavity

Theorem

Let $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$. Then, f is concave (convex) if and only if its restriction to every line segment in $\mathcal{U}$ is a concave (convex) function of one variable.

Concavity

Proof.

$\Leftarrow$

Let $x, y \in \mathcal{U}$ and $g(t) = f(tx + (1-t)y)$. By hypothesis, g is concave.

$$\begin{aligned} f(tx + (1-t)y) &= g(t) \\ &= g(t \times 1 + (1-t) \times 0) \\ &\geq tg(1) + (1-t)g(0) \\ &= tf(x) + (1-t)f(y) \end{aligned}$$

And then f is concave. $\square$

Concavity

Proof.

$\Rightarrow$

Let s_1 and s_2 in the domain of g. Let $t \in [0, 1]$.

$$\begin{aligned} g(ts_1 + (1-t)s_2) &= f((ts_1 + (1-t)s_2)x + (1 - (ts_1 + (1-t)s_2))y) \\ &= f(t(s_1x + (1-s_1)y) + (1-t)(s_2x + (1-s_2)y)) \\ &\geq tf(s_1x + (1-s_1)y) + (1-t)f(s_2x + (1-s_2)y) \\ &= tg(s1) + (1-t)g(s_2) \end{aligned}$$

And then g is concave. $\square$

Concavity

Theorem

Let f be a C^1 function on an interval $I \subset \mathbb{R}$. Then, f is concave on I if and only if:

$$f(y) - f(x) \leq f'(x)(y - x), \ \forall x, y \in I$$

f is convex if:

$$f(y) - f(x) \geq f'(x)(y - x), \ \forall x, y \in I$$

Concavity

Proof.

$\Rightarrow$

Let f be a concave function on I, $x, y \in I$ with $x \neq y$, and $t \in (0,1]$. Then,

$$
\begin{aligned}
tf(y) + (1-t)f(x) &\leq f(ty + (1-t)x) \\
f(y) - f(x) &\leq \frac{f(ty + (1-t)x) - f(x)}{t} \\
&= \frac{f(ty + (1-t)x) - f(x)}{t(y-x)}(y-x)
\end{aligned}
$$

Letting $t \to 0$ we have:

$$tf(y) + (1-t)f(x) \leq f'(x)(y-x)$$

$\square$

Concavity

Proof.

$\Leftarrow$

Let $f(y) - f(x) \leq f'(x)(x-y)$ $\forall x, y \in I$. Then,

$$\begin{aligned} f(x) - f((1-t)x + ty) &\leq f'((1-t)x+ty)(x - ((1-t)x+ty)) \\ &= -tf'((1-t)x+ty)(y-x) \end{aligned}$$

Equivalently,

$$f(y) - f((1-t)x+ty) \leq (1-t)f'((1-t)x+ty)(y-x)$$

Multiply the first by $(1-t)$ and the second by t, and adding up we get:

$$(1-t)f(x) + tf(y) \leq f((1-t)x+ty) \quad \textit{i.e. } f \text{ is concave.}$$

$\square$

Concavity

Theorem

Let f be a C^1 function on convex $\mathcal{U} \subseteq \mathbb{R}^n$. Then f is concave on $\mathcal{U}$ if and only if for all $x, y \in \mathcal{U}$:

$$f(y) - f(x) \leq Df(x)(y - x)$$

Similarly, f is convex on $\mathcal{U}$ if and only if for all $x, y \in \mathcal{U}$:

$$f(y) - f(x) \geq Df(x)(y - x)$$

Quasiconcavity

Definition

Quasiconcave and Quasiconvex function A function f defined on a convex $\mathcal{U} \subseteq \mathbb{R}^n$ is **quasiconcave** if for any $a \in \mathbb{R}$

$$C_a^+ \equiv \{x \in \mathcal{U} : f(x) \geq a\}$$

is a convex set.
Similarly, f is said to be **quasiconvex** if for any $a \in \mathbb{R}$

$$C_a^- \equiv \{x \in \mathcal{U} : f(x) \leq a\}$$

is a convex set.

Quasiconcavity

Theorem

Let f be a function defined on a convex set $\mathcal{U} \subseteq \mathbb{R}^n$. Then, the following statements are equivalent to each other:

1. *f is a quasiconcave function on $\mathcal{U}$.*
2. *$\forall x, y \in \mathcal{U}$ and $\forall t \in [0, 1]$,*

$$f(x) \geq f(y) \quad \Rightarrow \quad f(tx + (1-t)y) \geq f(y)$$

3. *$\forall x, y \in \mathcal{U}$ and $\forall t \in [0, 1]$,*

$$f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$$

Quasiconcavity

Suppose that f is a C^1 function on an open convex $\mathcal{U} \subseteq \mathbb{R}^n$. Then, f is quasiconcave on $\mathcal{U}$ if and only if:

$$f(y) \geq f(x) \quad \Rightarrow \quad Df(x)(y-x) \geq 0$$

f is quasiconvex on $\mathcal{U}$ if and only if

$$f(y) \leq f(x) \quad \Rightarrow \quad Df(x)(y-x) \leq 0$$

Quasiconcavity

Proof.

$\Rightarrow$

Let f be quasiconcave on $\mathcal{U}$ and that $f(y) \geq f(x)$ for some $x, y \in \mathcal{U}$. Then, $\forall t \in [0,1]$

$$f(x + t(y-x)) \geq f(x)$$

Since

$$\frac{f(x + t(y-x)) - f(x)}{t} \geq 0$$

and $\forall t \in (0,1)$, we multiply by $\frac{(y-x)}{(y-x)}$ and let $t \to 0$ to get

$$Df(x)(y-x) \geq 0$$

$\square$